Trinity College

Semester One Examination, 2017

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 3

Section One: Calculator-free

SOLUTIONS

Student Number: In figures

In words ________________________________

Your name ________________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	11	11	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

The position vector of R, the centre of a sphere with diameter $\overrightarrow{PQ}$, is $2\mathbf{i} - \mathbf{k}$ and the position vector of Q is $8\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

(a) Determine the position vector of P. (2 marks)

Solution
$\overrightarrow{QR} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 8 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \\ -2 \end{pmatrix}$ $P = R + \overrightarrow{QR} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} -6 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ -3 \end{pmatrix}$
Specific behaviours
✓ determines $\overrightarrow{QR}$ ✓ determines position vector

(b) Determine the vector equation of the sphere. (2 marks)

Solution
$r = \lvert\overrightarrow{QR}\rvert = \sqrt{36 + 9 + 4} = 7$ $\left\lvert \mathbf{r} - \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right\rvert = 7$
Specific behaviours
✓ determines radius   ✓ writes in correct form

(c) The sphere intersects the y-axis where $y = a$. Determine the value(s) of the constant a. (2 marks)

Solution
$\left\lvert \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right\rvert = 7 \Rightarrow a^2 + 4 + 1 = 49$ $a = \pm 2\sqrt{11}$
Specific behaviours
✓ substitutes point into equation and expands   ✓ states both values

Question 2 **(5 marks)**

A function is defined by $g(z) = 2z^4 - z^3 + 7z^2 - 4z - 4$.

(a) Show that $z = 1$ and $z = 2i$ are both zeros of $g(z)$. (2 marks)

Solution
$g(1) = 2 - 1 + 7 - 4 - 4 = 0$ $g(2i) = 2(2i)^4 - (2i)^3 + 7(2i)^2 - 4(2i) - 4$ $= 32 + 8i - 28 - 8i - 4 = 0$
Specific behaviours
✓ shows all terms of $g(1)$ and that they sum to zero ✓ substitutes $2i$ correctly, shows simplified terms of $g(2i)$ and that they sum to zero

(b) Determine all solutions to $g(z) = 0$. (3 marks)

Solution
$2z^4 - z^3 + 7z^2 - 4z - 4 = (z-1)(z-2i)(z+2i)(2z+a)$ $= (z-1)(z^2+4)(2z+a)$ By inspection, $a = 1$ Hence $g(z) = (z-1)(z-2i)(z+2i)(2z+1)$ $g(z) = 0$ when $z = 1, z = 2i, z = -2i, z = -\frac{1}{2}$.
Specific behaviours
✓ shows three factors of $g(z)$ and form of fourth ✓ determines fourth factor ✓ lists all solutions

Question 3 **(8 marks)**

Simplify the following into the form $x + iy$.

(a) $\dfrac{3}{2i} + 2i$. (2 marks)

Solution
$\dfrac{3}{2i} \times \dfrac{i}{i} = -\dfrac{3i}{2}$ $2i - \dfrac{3}{2}i = \dfrac{1}{2}i$
Specific behaviours
✓ makes denominator of fraction real   ✓ simplifies into required form

(b) $\dfrac{1}{(2-i)^2}$. (3 marks)

Solution
$\dfrac{1}{(2-i)^2} = \dfrac{1}{3-4i}$ $= \dfrac{1}{3-4i} \times \dfrac{3+4i}{3+4i}$ $= \dfrac{3+4i}{25} = \dfrac{3}{25} + \dfrac{4}{25}i$
Specific behaviours
✓ expands denominator   ✓ real denominator   ✓ simplifies into required form

(c) $\left(-\sqrt{2} + \sqrt{2}i\right)^6$. (3 marks)

Solution
$\left(-\sqrt{2} + \sqrt{2}i\right)^6 = \left(2\,\text{cis}\left(\dfrac{3\pi}{4}\right)\right)^6$ $= 64\,\text{cis}\left(\dfrac{18\pi}{4}\right)$ $= 64\,\text{cis}\dfrac{\pi}{2}$ $= 64i$
Specific behaviours
✓ converts to polar form   ✓ applies de Moivre's theorem   ✓ simplifies into required form

Question 4 **(7 marks)**

The function f is defined by $f(x)=\frac{1}{1-x}$.

(a) Evaluate $f(f(-1))$. (1 mark)

Solution
$f(-1)=\frac{1}{2}, f\left(\frac{1}{2}\right)=2$
Specific behaviours
✓ correct value

(b) Determine and simplify an expression for $f \circ f(x)$. (2 marks)

Solution
$$\begin{aligned} f(f(x)) &= \frac{1}{1-\frac{1}{1-x}} \\ &= 1 \div \frac{1-x-1}{1-x} \\ &= \frac{1-x}{-x} \\ &= \frac{x-1}{x} = 1-\frac{1}{x} \end{aligned}$$
Specific behaviours
✓ creates composite function ✓ simplifies with positive denominator

(c) For $f \circ f(x)$, state the

(i) domain. (2 marks)

Solution
$D_{f\circ f}=\{x: x\in\mathbb{R}, x\neq 0, x\neq 1\}$
Specific behaviours
✓ states $x\neq 0$ ✓ states $x\neq 1$

(ii) range. (2 marks)

Solution
$R_{f\circ f}=\{y: y\in\mathbb{R}, y\neq 0, y\neq 1\}$
Specific behaviours
✓ states $y\neq 0$ ✓ states $y\neq 1$

Question 5 **(7 marks)**

(a) The equation $2z^2 + 3z + 5 = 0$ has roots of α and β. Determine the value of

(i) $\alpha + \beta$. (1 mark)

Solution
$2\left(z^2 + \frac{3}{2}z + \frac{5}{2}\right) = 0 \Rightarrow \alpha + \beta = -\frac{3}{2}$
Specific behaviours
✓ states value

(ii) $\alpha\beta$. (1 mark)

Solution
$\alpha\beta = \frac{5}{2}$
Specific behaviours
✓ states value

(iii) $2\alpha^2 + 3\alpha + 5$. (1 mark)

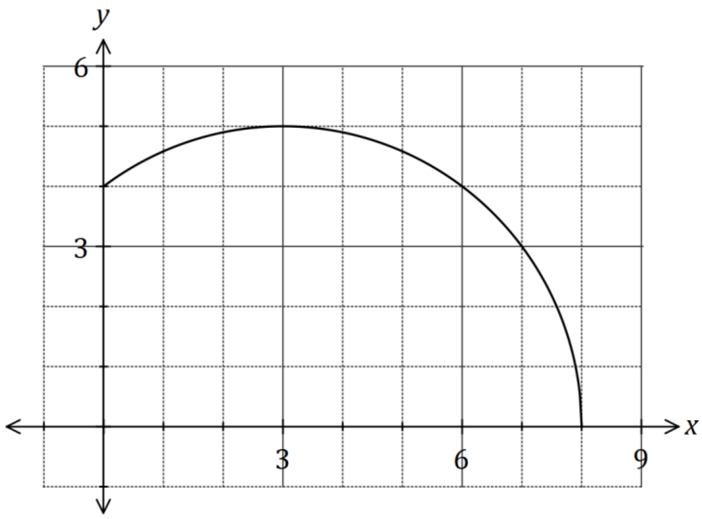
Solution
$2\alpha^2 + 3\alpha + 5 = 0$
Specific behaviours
✓ states value

(b) Determine the values of the real constants a and b if $z - 2 + i$ is a factor of $z^3 + az + b$. (4 marks)

Solution
$z = 2 - i$ is a root so $(2-i)^3 + a(2-i) + b = 0$ $((3-4i)(2-i) + 2a - ai + b = 0$ $2 - 11i + 2a - ai + b = 0$ Equate imaginary parts: $a = -11$ Equate real parts: $2 + 2(-11) + b = 0 \Rightarrow b = 20$ $a = -11, b = 20$
Specific behaviours
✓ identifies roots and uses factor theorem ✓ expands correctly ✓ equates real and imaginary parts ✓ correct values

Question 6 **(6 marks)**

Let $f(x) = \sqrt{16 + 6x - x^2},\ 0 \le x \le 8$. The graph of $y = f(x)$ is shown below.



(a) In order that $y = f^{-1}(x)$ is a function, the domain of f must be restricted to $k \le x \le 8$. Explain why this restriction is necessary and state the minimum value of k. (2 marks)

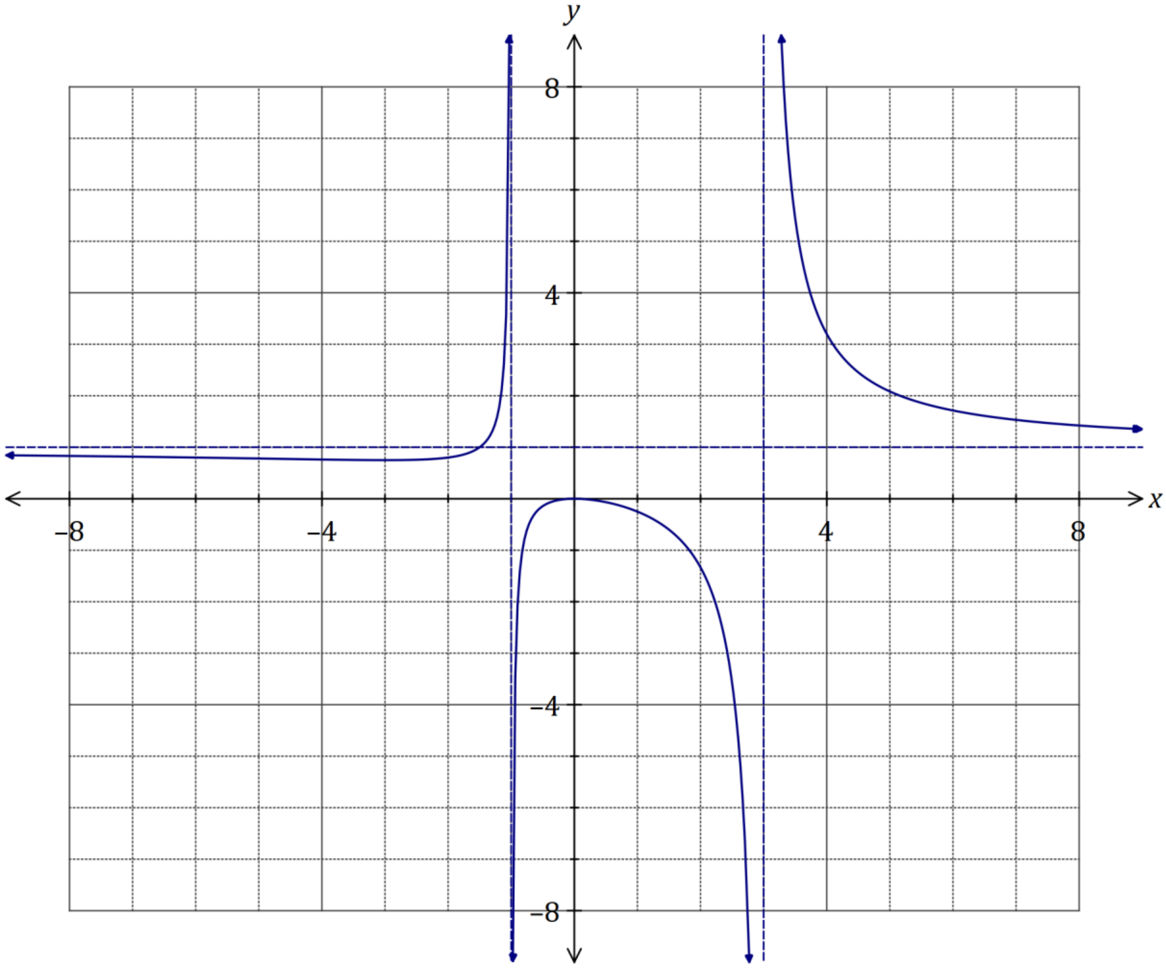
Solution
A one-to-one relationship must exist for $f^{-1}(x)$ to be a function. The minimum value of k is 3.
Specific behaviours
✓ require one-to-one relationship ✓ value of k

(b) Using the restriction from (a), determine the inverse function of f and its domain. (4 marks)

Solution
Let $y^2 = 16 + 6x - x^2$ $x^2 - 6x = 16 - y^2$ $(x-3)^2 = 25 - y^2$ $x = 3 + \sqrt{25 - y^2}$ $y = f^{-1}(x) = 3 + \sqrt{25 - x^2}, \qquad 0 \le x \le 5$
Specific behaviours
✓ square and isolate x terms ✓ completes square and solves for x ✓ expresses as function of x ✓ states domain

Question 7 **(6 marks)**

On the axes below, draw the graph of $y = \frac{x^2}{x^2-2x-3}$, clearly showing key features and the behaviour of the curve near the asymptotes.



Solution
See graph. $x \to \pm\infty, y \to 1$ $x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1) = 0$ $\frac{dy}{dx} = \frac{2x(x^2-2x-3)-x^2(2x-2)}{(x^2-2x-3)^2}$ $\frac{dy}{dx} = 0 \Rightarrow 2x(x^2-2x-3) - 2x(x^2-x) = 0 \Rightarrow 2x(-x-3) = 0$ Stationary points at $(0, 0)$ and $\left(-3, \frac{3}{4}\right)$
Specific behaviours
✓ identifies vertical asymptotes $x = -1$ and $x = 3$ ✓ curve approaches vertical asymptotes correctly ✓ identifies horizontal asymptote $y = 1$ ✓ curve approaches horizontal asymptote correctly as $x \to \infty$ and $x \to -\infty$ ✓ identifies maximum at $(0, 0)$ ✓ identifies minimum at $\left(-3, \frac{3}{4}\right)$

Question 8 **(7 marks)**

(a) Two of the solutions to the equation $z^n = 1, n \in \mathbb{Z}^+$, are $z = \text{cis}\frac{\pi}{2}$ and $z = \text{cis}\frac{\pi}{3}$.

(i) State another solution to the equation. (1 mark)

Solution
$z = 1$ (or any of form $z = \frac{\pi}{6k}, k \in \mathbb{Z}$)
Specific behaviours
✓ states valid solution

(ii) Determine, with reasons, the minimum value of n. (3 marks)

Solution
$\arg z = \dfrac{\pi}{2} \Rightarrow n = 4k$ and $\arg z = \dfrac{\pi}{3} \Rightarrow n = 6k, k \in \mathbb{Z}$ LCM of 4, 6 is 12 Minimum value of $n = 12$
Specific behaviours
✓ indicates n is multiple of 4 ✓ indicates n is multiple of 6 ✓ deduces minimum value of n

(b) If $z = \text{cis}\frac{\pi}{4}$, determine the sum of the geometric series $1 + z + z^2 + z^3 + ... + z^{24}$. Explain your answer. (3 marks)

Solution
$\lvert z \rvert = 1$ and so $z^0 + z^1 + z^2 + z^3 + ... + z^7 = 0$ Likewise, every consecutive 8 terms of sequence. Hence $1 + z + z^2 + z^3 + ... + z^{23} + z^{24} = z^{24} = \text{cis}\, 0 = 1$
Specific behaviours
✓ indicates sum of first 8 consecutive terms is zero ✓ indicates sum of any 8 consecutive terms is zero ✓ determines sum

End of questions

Additional working space

Question number: _________